Fast and Robust Bootstrap[1]

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[1] Main Reference: Salibian-Barrera and Zamar (2002)

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Motivation

- Numerical stability
- Computational cost

Formulation of the Problem

- **Data:** $(y_i, \mathrm{z}_i')', (y_2, \mathrm{z}_2')', \ldots (y_n, \mathrm{z}_n')'$
- $(y_1, \mathrm{z}_1')' \overset{iid}{\sim} H$, let $x_i = (1, z_i')' \in \mathbb{R}^p$, and the model is,

$$y_i = x_i'\beta_0 + \sigma_0\varepsilon_i, \;\; i = 1, 2, ..., n \tag{1}$$

- **Ideal Situation:** y_i and z_i are independently distributed for all i.
  $y_i \sim F_0, z_i \sim G_0, (y_i, z_i')' \sim H_0$.
- $H \in \mathscr{H}_\varepsilon = \{H = (1-\varepsilon)H_0 + \varepsilon H^*\}$,where H^* is an arbitrary and unspecified function and $0 \leq \varepsilon < 1/2$.

MM-Estimation

- We consider MM-estimation which is based on two loss functions ρ_0 and ρ_1, say.
- If $\hat{\beta}_n$ is the MM-estimate of β, then it satisfies the following equations,

$$\frac{1}{n}\sum_{i=1}^{n}\rho_1^{'}\left(\frac{y_i - x_i'\hat{\beta}_n}{\hat{\sigma}_n}\right)x_i = 0 \tag{2}$$

- $\hat{\sigma}_n$ is scale S-estimate which minimizes the following equation,

$$\frac{1}{n}\sum_{i=1}^{n}\rho_0\left(\frac{y_i - x_i'\beta}{\hat{\sigma}_n(\beta)}\right) = b \tag{3}$$

- $\tilde{\beta}_n$ is the associated S-regression estimate.

Fast Bootstrap

- $\hat{\beta}_n$ can be represented as a solution of fixed point equations:

$$\hat{\beta}_n = f_n(\hat{\beta}_n)$$

- Here f_n depends on the observed data $\{(y_i, x_i), i = 1, \ldots, n\}$ and for given data f_n is given by,

$$f_n(\hat{\beta}_n) = \left[\sum_{i=1}^{n} w_i(\hat{\beta}_n) x_i x_i' \right]^{-1} \sum_{i=1}^{n} w_i(\hat{\beta}_n) x_i y_i \qquad (4)$$

$w_i(\hat{\beta}_n) = \frac{\rho_1'(r_i/\hat{\sigma}_n)}{r_i}$, where $r_i = y_i - \hat{\beta}_n' x_i$.

- Given a bootstrap sample $\{(y_i^*, x_i^*), i = 1, \ldots, n\}$ the recalculated estimates $\hat{\beta}_n^b$ solves, $\hat{\beta}_n^b = f_n^*(\hat{\beta}_n^b)$.

Fast Bootstrap

- f_n^* has the same form of f_n, except it is based on the bootstrap samples and the corresponding weights are, given by,

$$w_i^*(\hat{\beta}_n^b) = \frac{\rho_1'(r_i^b/\hat{\sigma}_n)}{r_i}, \quad \text{where } r_i^b = y_i^* - \hat{\beta}_n'^b x_i^*.$$

- Instead of computing $\hat{\beta}_n^b$ we consider, $\hat{\beta}_n^* = f_n^*(\hat{\beta}_n)$, i.e. in f_n^* we use the weights as,

$$w_i^*(\hat{\beta}_n) = \frac{\rho_1'(r_i^*/\hat{\sigma}_n)}{r_i}, \quad \text{where } r_i^* = y_i^* - \hat{\beta}_n' x_i^*.$$

- It can be shown that $\hat{\sigma}_n$ has a weighted average representation and so it is possible to define $\hat{\sigma}_n^*$ for the bootstrap samples similarly.
- $\hat{\beta}_n^*$ may not reflect true variability of $\hat{\beta}_n$, on applying correction factor our final estimate is:

$$\hat{\beta}_n^{R*} - \hat{\beta}_n = M_n(\hat{\beta}_n^* - \hat{\beta}_n) + d_n(\hat{\sigma}_n^* - \hat{\sigma}_n)$$

Asymptotic Properties of Fast Bootstrap

- Now that we have put the forward the methodology, we focus on the asymptotic properties of Fast bootstrap estimates.
- The next theorem will show that the asymptotic distribution of fast bootstrap is the same as that of MM-regression estimator.
- We proceed with stating a few regularity conditions on the form of ρ_0 and ρ_1 defined earlier.

Some conditions

MM-estimates are based on two loss function $\rho_0 : \mathbb{R} \to \mathbb{R}_+$ and $\rho_1 : \mathbb{R} \to \mathbb{R}_+$ (defined earlier) which determine the breakdown point and the efficiency of the estimate. They satisfy the following conditions:

C1 $\forall u \in \mathbb{R}$, $\rho_0(-u) = \rho_0(u)$ and $\rho_1(u) = \rho_1(-u)$;

C2 $\rho_0(0) = 0 = \rho_1(0)$;

C3 ρ_0 and ρ_1 are continuously differentiable functions;

C4 $\sup_x \rho_0(x) = \sup_x \rho_1(x) = 1$;

C5 If $\rho_0(u) < 1$ and $0 \leq v < u$, then $\rho_0(v) < \rho_0(u)$. Same condition holds for ρ_1.

Some established results

Salibian-Barrera and Zamar (2002) proved that that $\hat{\beta}_n$ (MM-regression estimator), $\hat{\sigma}_n$ (S-scale estimator) and $\tilde{\beta}_n$ (S-regression estimator) are consistent (weakly) for true values $\beta,\ \sigma,\ \&\ \tilde{\beta}$ where,

$$\mathbb{E}[\rho_1'((Y - X'\beta)/\sigma)] = 0$$
$$\mathbb{E}[\rho_0((Y - X'\tilde{\beta})/\sigma)] = b$$
$$\mathbb{E}[\rho_0'((Y - X'\tilde{\beta})/\sigma)] = 0$$

This result is essential in stating the first main theorem of this topic.

Convergence of Fast Robust Distribution

Theorem

If ρ_0 and ρ_1 satisfies the conditions (C1-C5) and have continuous third order derivatives, then given the consistency of $\hat{\beta}_n$, $\hat{\sigma}_n$ and $\tilde{\beta}$, and under a few regularity conditions, almost all sample sequences $\sqrt{n}(\hat{\beta}_n^{R*} - \hat{\beta}_n)$ converges weakly, as n goes to infinity, to the same limit distribution as $\sqrt{n}(\hat{\beta}_n - \beta)$.

Robustness of Fast Bootstrap

- We now focus on the robustness properties of our fast bootstrap.
- Let q_t be the t^{th} upper quantile of a statistics $\hat{\theta}_n$ i.e. q_t satisfies

$$P[\hat{\theta}_n > q_t] = t$$

- Singh (1998) defines upper breakdown point of a quantile estimate $\hat{q}_t$ as the minimum proportion of asymmetric contamination that can drive it over any finite bound.
- An estimator based on bootstrap sample can potentially break down if the expected proportion of bootstrap samples that contain more outliers than the breakdown point of the estimate (say τ^*) to be more than t.

Breakdown point of the fast bootstrap quantiles

Theorem

Let $(y_1, x_1')', \ldots, (y_n, x_n')' \in \mathbb{R}^{p+1}$ be the random sample following linear model. Assume that the explanatory variables $x_1, \ldots, x_n$ in $\mathbb{R}^p$ are in general position. Let $\hat{\beta}_n$ be an MM-regression estimate and let ε^* be its breakdown point. Then the breakdown point of the t^{th} fast bootstrap quantile estimate of the regression parameters β_j, $j = 1, \ldots, p$ is given by $\min(\varepsilon^*, \varepsilon_R)$, where ε_R satisfies

$$\varepsilon_R = \inf\{\delta \in [0,1] : P[\texttt{Binomial}(n,\delta) > n - p] \geq t\}$$

Singh (1998) obtained the upper breakdown point of bootstrap estimate $\hat{q}_t$ of q_t:

$$\varepsilon_C = \inf\{\delta \in [0,1] : P[\texttt{Binomial}(n,\delta) \geq [\varepsilon^* n]] \geq t\}$$

If $n > 2p$, then $[\varepsilon^* n] \leq [n/2] < n - p$. Thus we can clearly see that $\varepsilon_C < \varepsilon_R$.

Simulation Study

Data Description

- Generated the data $y_i = \beta_0 + \beta_1 x_i + \varepsilon,\ i = 1, \ldots, n$ for $n = 30$ and 100.

- $x_i \sim Normal(0,1),\ \beta_0 = 5$ and $\beta_1 = 5$.

- The errors are generated from F_ε with,

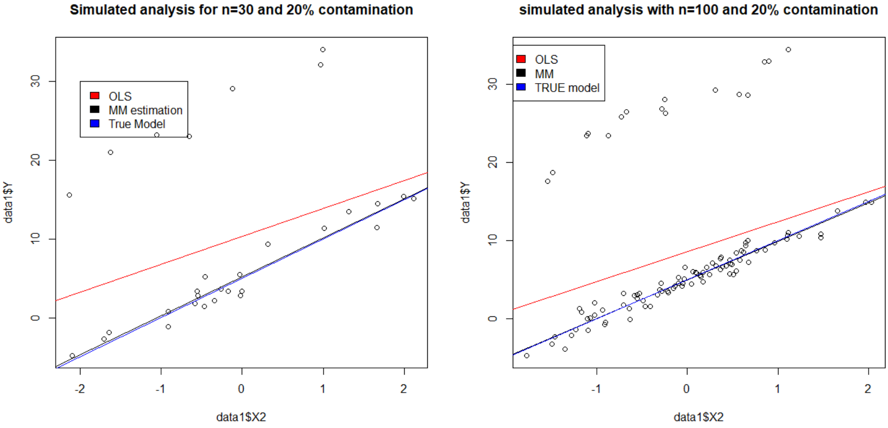
$$F_\varepsilon(x) = (1-\varepsilon)\Phi(x) + \varepsilon F_u(x)$$

  Φ is the CDF of $Normal(0,1)$andF_u is the CDF of $Uniform(20,25)$

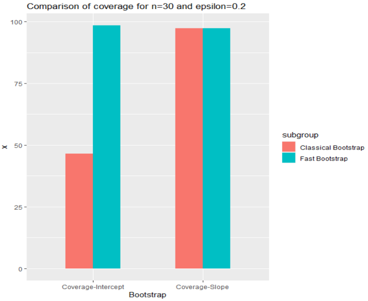
- Considered $\varepsilon = 0.0, 0.20$, i.e. considered 0% and 20% contamination in the error distribution.

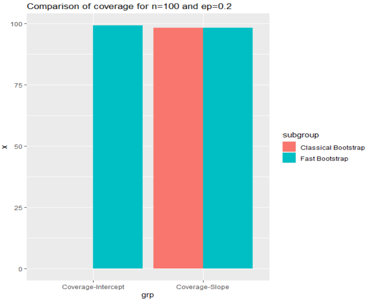
- Generated 1000 datasets from the above distribution and built 99% confidence intervals for the parameters

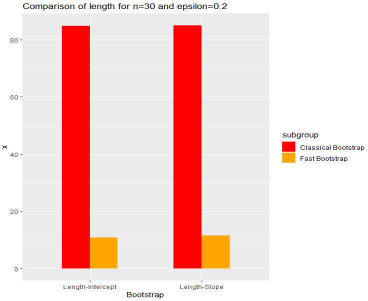
Robustness regression fits

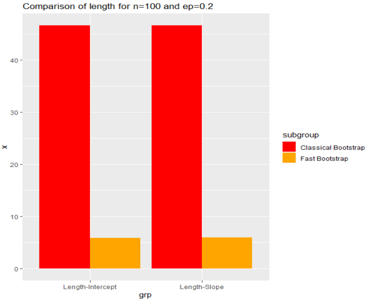


Numerical stability results

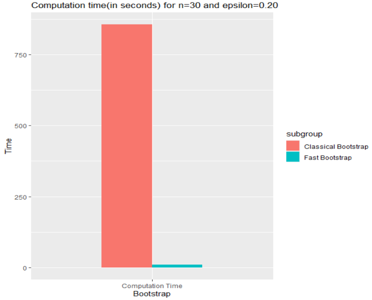


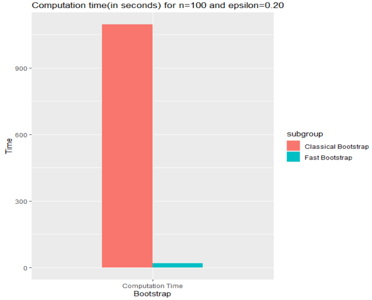


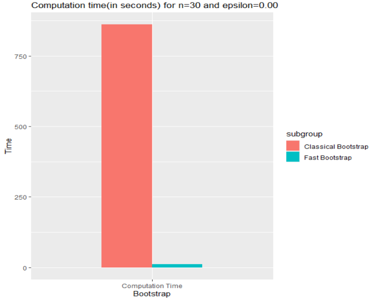


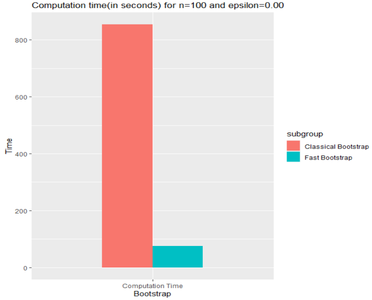


Computational cost results

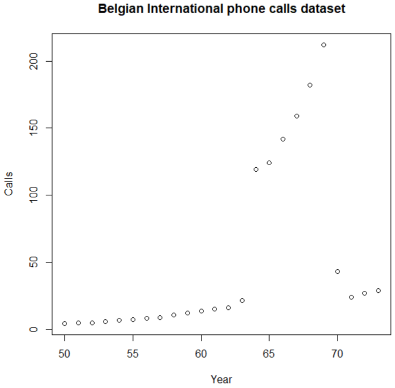






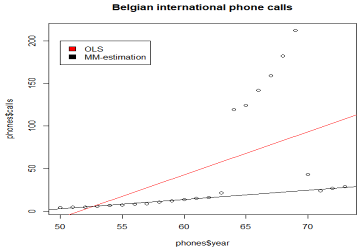


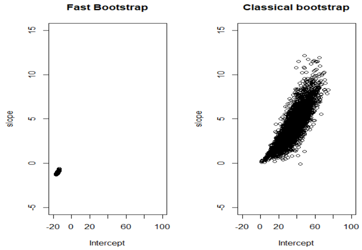
1)Belgian International Phone Calls[2]
Using 10000 fast bootstrap calculations we estimate the distribution of robust regression estimates and compare results with classical bootstrap method.

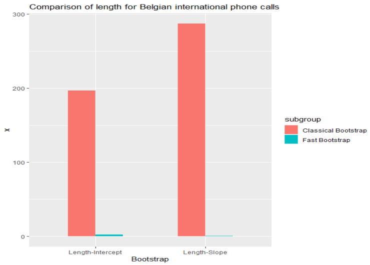


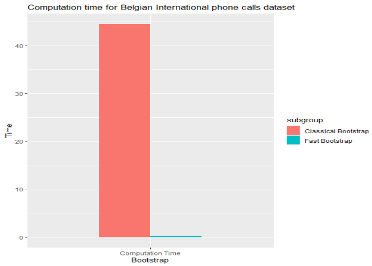
[2] MASS Package in R

Data Analysis-II



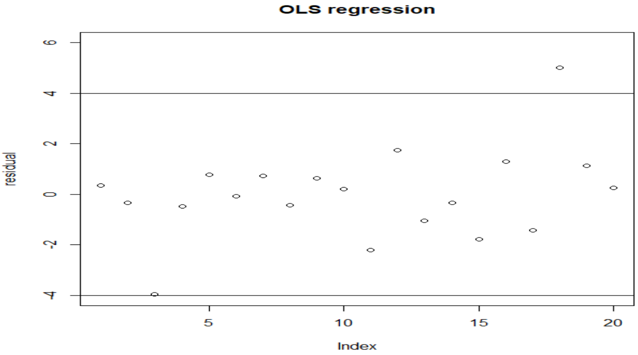






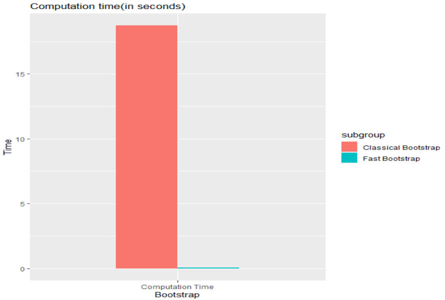
2)Verbal test score data[3]

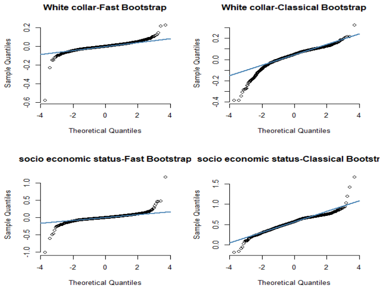
The data consist of verbal mean test scores from 20 schools.There are 5 explanatory variables.The plot of residuals below confirms presence of outliers.

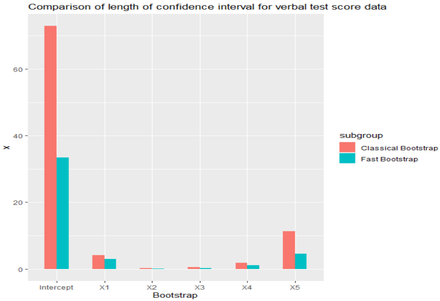


[3]Coleman et. al (1966)

Results on Verbal test score data







Reference

Salibian-Barrera, M. and Zamar, R. H. (2002). Bootstrapping robust estimates of regression. *Annals of Statistics*, pages 556–582.

Singh, K. (1998). Breakdown theory for bootstrap quantiles. *The Annals of Statistics*, 26(5):1719 – 1732.